# What Representational Similarity Measures Imply about Decodable Information


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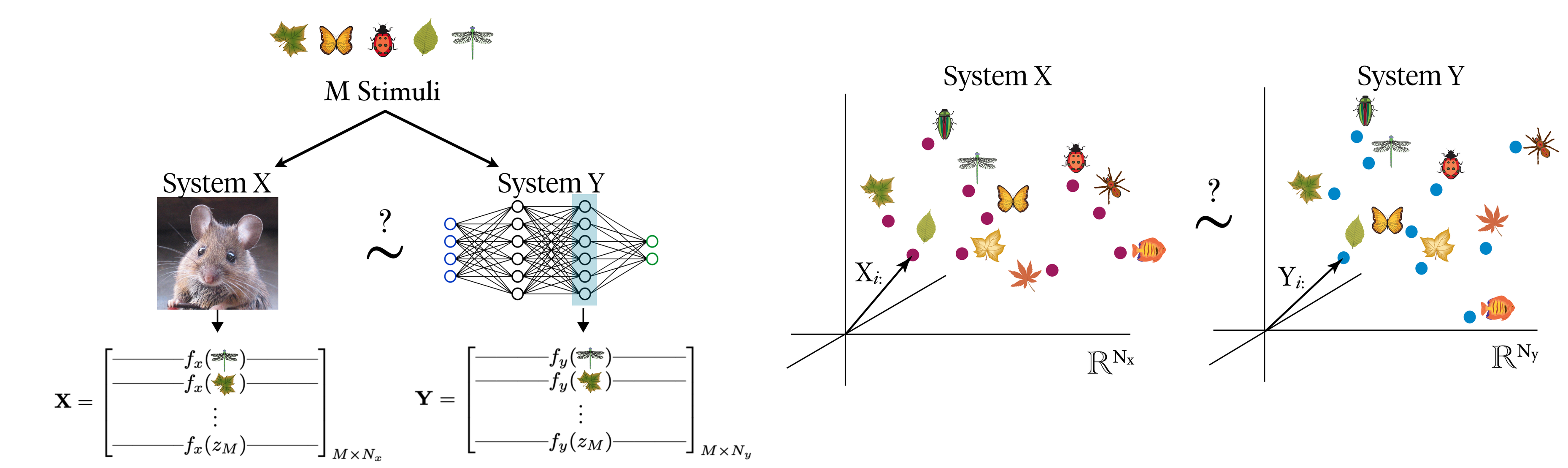
## Measuring Representational Similarity

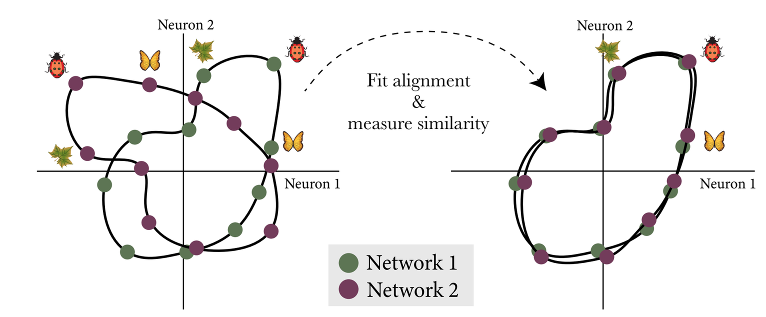
**Comparative analyses are important tools for understanding complex systems**

How do we quantify similarity between ***neural representations***?




## Many Existing Approaches

**(Dis)similarity measures that transform or align neural dimensions**



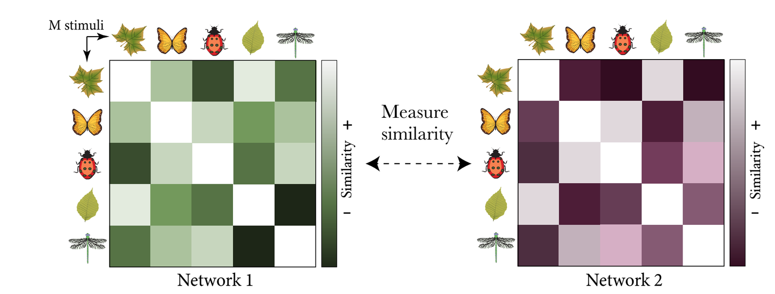

- Linear regression [1]
- Canonical correlations analysis [2]
- **Procrustes shape distances** [3]

$$\mathscr{P}(X,Y) = \min_{\mathbf{O} \in \mathcal{O}(N)} \|\mathbf{X} - \mathbf{Y}\mathbf{O}\|_F$$

*Measure Euclidean distance between representations after rotational alignment*

*Takes values from 0 to ∞ (distance)*

**(Dis)similarity measures that quantify stimulus-by-stimulus relationships**




- Representational Similarity Analysis (RSA) [4]
- Bures distance (Muzellec et. al. 2018) [6]
- **Centered Kernel Alignment (CKA)** [5]

$$CKA(\mathbf{X},\mathbf{Y}) = \frac{\mathrm{Tr}\, \mathbf{X}\mathbf{X}^\top \mathbf{Y}\mathbf{Y}^\top}{\sqrt{\mathrm{Tr}\,(\mathbf{X}\mathbf{X}^\top)^2\, \mathrm{Tr}\,(\mathbf{Y}\mathbf{Y}^\top)^2}}$$

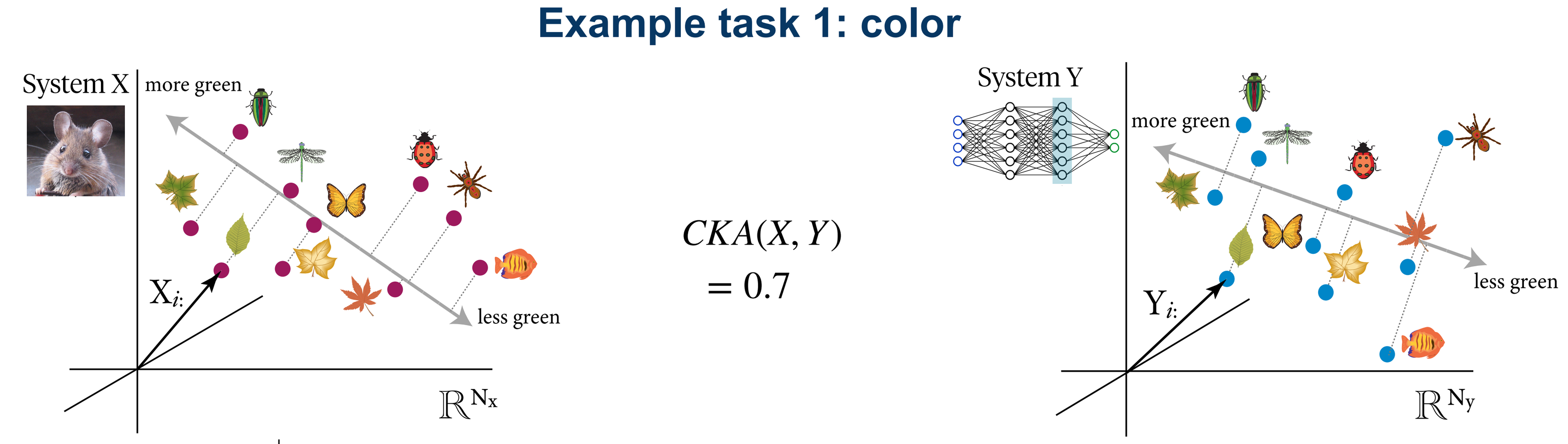
*(Linear kernels)*

*Compute inner product between centered representations and normalize*

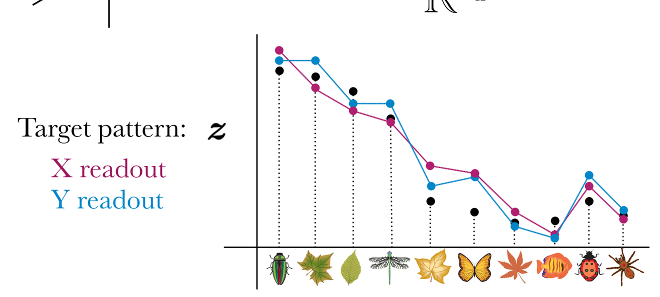
*Takes values from 0 to 1 (similarity)*

## Comparing representations with linear decoding

### Example task 1: color




$CKA(X,Y)$
$= 0.7$

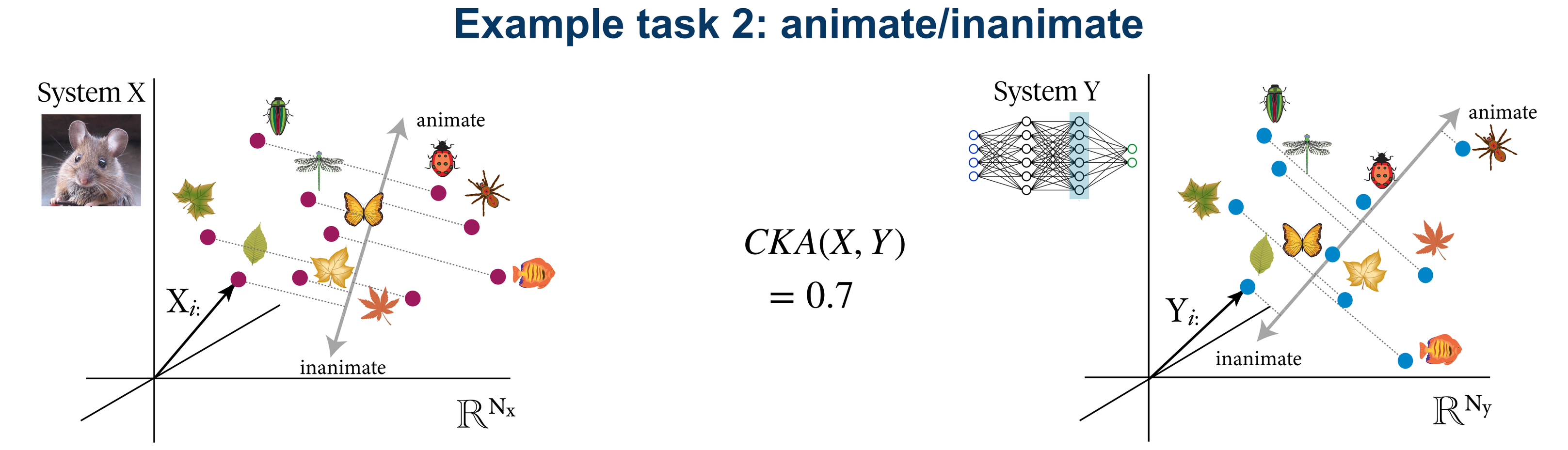



Compute optimal weights for system $X$
$\Longrightarrow$ Decoded pattern: $\boldsymbol{X}\boldsymbol{w}^*$

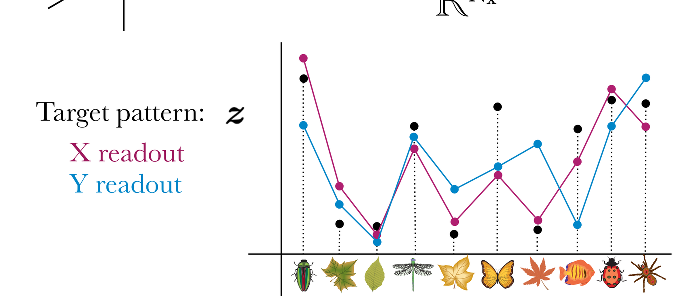
Compute optimal weights for system $Y$
$\Longrightarrow$ Decoded pattern: $\boldsymbol{Y}\boldsymbol{v}^*$

Measure similarity:
(Scaled to be unit vectors)
$\langle \boldsymbol{X}\boldsymbol{w}^*, \boldsymbol{Y}\boldsymbol{v}^* \rangle$
$= 0.82$

### Example task 2: animate/inanimate




$CKA(X,Y)$
$= 0.7$




Compute optimal weights for system $X$
$\Longrightarrow$ Decoded pattern: $\boldsymbol{X}\boldsymbol{w}^*$

Compute optimal weights for system $Y$
$\Longrightarrow$ Decoded pattern: $\boldsymbol{Y}\boldsymbol{v}^*$

Measure similarity:
(Scaled to be unit vectors)
$\langle \boldsymbol{X}\boldsymbol{w}^*, \boldsymbol{Y}\boldsymbol{v}^* \rangle$
$= 0.63$

## Average decoding similarity/distance

★ Similarity depends on the choice of decoding task

**Idea**: Measure similarity over an ensemble of decoding tasks

*average decoding similarity (ADS)*

$$\mathbb{E}_{z \sim P_z} \langle \boldsymbol{X}\boldsymbol{w}^*, \boldsymbol{Y}\boldsymbol{v}^* \rangle$$

*average decoding distance (ADD)*

$$\mathbb{E}_{z \sim P_z} \|\boldsymbol{X}\boldsymbol{w}^* - \boldsymbol{Y}\boldsymbol{v}^*\|_2^2$$

To compute these, we must choose:

1. Regression loss function
2. Ensemble of tasks to average over

**We show:** certain choices here $\Longrightarrow$ average decoding similarity/distance = popular representational similarity/distance measures

## General mathematical framework

➡ Set up a family of linear decoding problems

Decoding optimization problem:

$$\underset{\boldsymbol{w}}{\text{maximize}} \quad \underbrace{\tfrac{1}{M}\boldsymbol{z}^\top \boldsymbol{X}\boldsymbol{w}}_{\substack{\text{Maximize overlap} \\ \text{between } \mathbf{Xw} \text{ and } \mathbf{z}}} - \underbrace{\tfrac{1}{2}\boldsymbol{w}^\top \boldsymbol{G}(\boldsymbol{X})\boldsymbol{w}}_{\substack{\text{Penalty on a norm of} \\ \text{the weights } \mathbf{w}}}$$

This problem has a nice closed form solution: $\boldsymbol{w}^* = \frac{1}{M}\boldsymbol{G}(\boldsymbol{X})^{-1}\boldsymbol{X}^\top \boldsymbol{z}$ *Optimal Decoding Weights*

$\mathbf{G}(\cdot)$ is a function mapping $\mathbb{R}^{M \times N} \to$ symmetric positive definite $N \times N$ matrices

Consider $\boldsymbol{G}(\boldsymbol{X}) = a\boldsymbol{C}_X + b\boldsymbol{I}$ with neuron-by-neuron covariance $\boldsymbol{C}_X := \frac{1}{M}\boldsymbol{X}^\top \boldsymbol{X}$

## Relations to geometric similarity measures

### Special cases

Take $a = 1$, $b = \lambda$ ➡

$$\mathbf{w}^* = \text{argmin } \|\mathbf{X}\mathbf{w} - \mathbf{z}\|_2^2 + \lambda \mathbf{I}$$
$$\mathbf{v}^* = \text{argmin } \|\mathbf{Y}\mathbf{w} - \mathbf{z}\|_2^2 + \lambda \mathbf{I}$$

*Ridge regression*

If we also assume identity covariance structure of the task ensemble we are averaging over, i.e. $\mathbb{E}_{\mathbf{z}}\left[\mathbf{z}\mathbf{z}^\top\right] = \mathbf{I}$

Take $a = 0$, $b = 1$ ➡

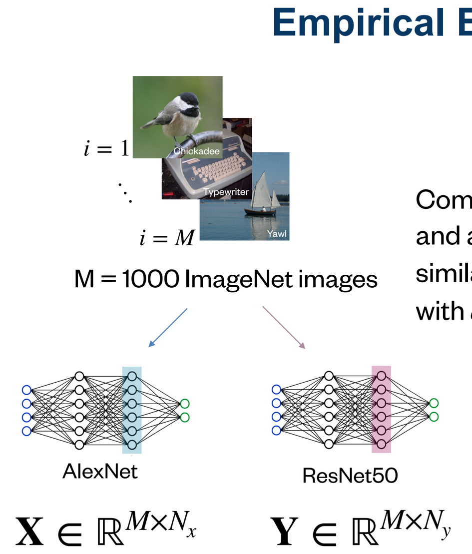
$$\frac{\mathbb{E}\langle \boldsymbol{X}\boldsymbol{w}^*, \boldsymbol{Y}\boldsymbol{v}^* \rangle}{\sqrt{\mathbb{E}\langle \boldsymbol{X}\boldsymbol{w}^*, \boldsymbol{X}\boldsymbol{w}^* \rangle \mathbb{E}\langle \boldsymbol{Y}\boldsymbol{v}^*, \boldsymbol{Y}\boldsymbol{v}^* \rangle}} = \mathrm{CKA}(\boldsymbol{X}, \boldsymbol{Y})$$

*(Normalized) Average decoding similarity*

**More in paper:**

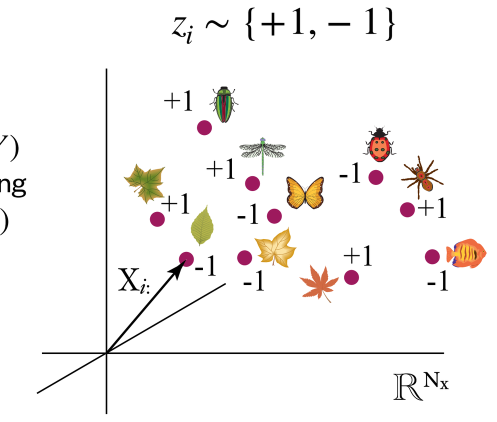
| Similarity measure | $a$ | $b$ |
|---|---|---|
| Linear CKA | 0 | $b$ |
| GULP | 1 | $\lambda$ |
| CCA | 1 | 0 |
| ENSD | 0 | $\frac{1}{M}\frac{\mathrm{Tr}[\boldsymbol{C}_X^2]}{\mathrm{Tr}[\boldsymbol{C}_X]}$ |

### Empirical Example: CKA

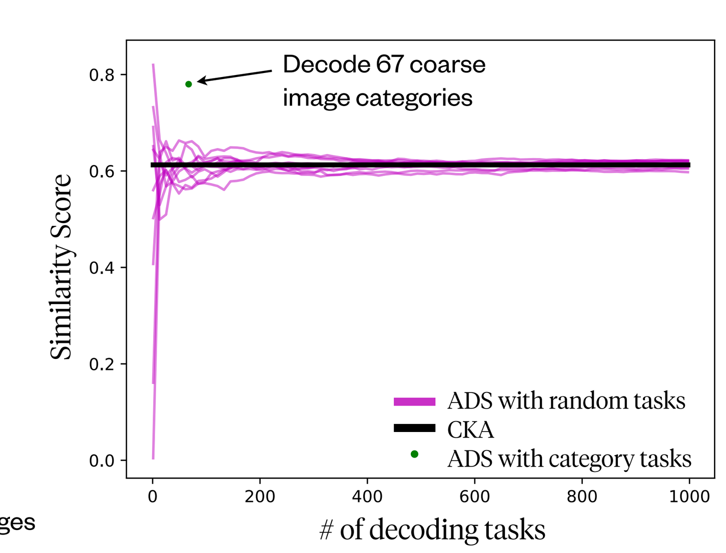



Compute $CKA(X,Y)$ and average decoding similarity $ADS(X,Y)$ with $a = 0, b = 1$

$z_i \sim \{+1, -1\}$

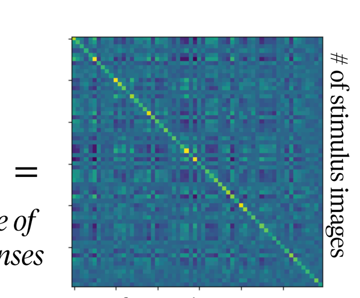


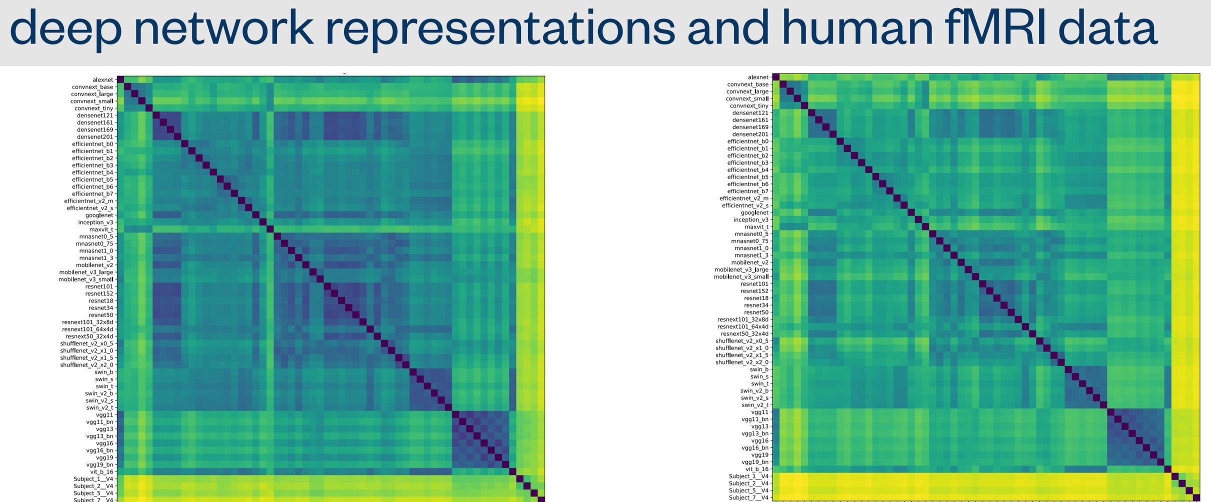

Ensemble of decoding tasks:
‣ Random binary partitions of input images




## Example: Comparing deep network representations and human fMRI data

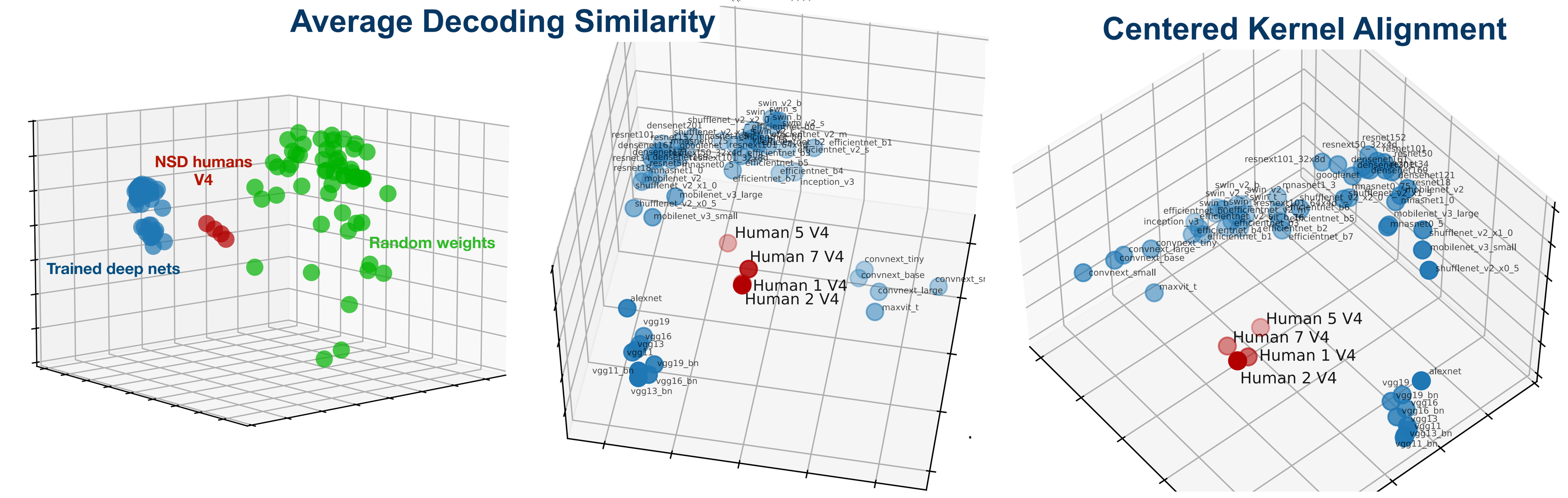
$\mathbb{E}_{\mathbf{z}}\left[\mathbf{z}\mathbf{z}^\top\right] =$
*Covariance of fMRI responses*



### Average Decoding Similarity

### Centered Kernel Alignment




## Links

**Paper:**
arXiv:2411.08197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## References


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6. Muzellec, B., and Cuturi, M. "Generalizing Point Embeddings using the Wasserstein Space of Elliptical Distributions". NeurIPS (2018).